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A Historical Note on Sturmian Theory*

WILLIAM T. REID†

Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73069

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1. INTRODUCTION

At the end of his famous 1836 memoir [11], Sturm states that the presented theory of real linear homogeneous second-order differential equations corresponds to an entirely analogous theory that he had developed earlier for second-order linear difference equations, and that it was in the course of considering solutions of the linear difference equations that he discovered his theorem on the number of real roots of a real polynomial on an interval of the real line.

The work of Sturm on linear difference equations was lost, and from comments of Porter [6] and Bôcher [1, 2] the general consensus seems to be that Sturm considered his oscillation and comparison theorems for ordinary differential equations to be analogous to results he had obtained for solutions of difference equations, and that their interrelations were heuristic rather than precise in nature.

For the present author the relationship between differential and difference equations remained in the domain of analogy until his study of generalized differential equations [8], and his appreciation of the connection between the Sturmian theory for real second-order differential equations and Sturm's theorem on the zeros of polynomials remained nebulous until his considera-tion of generalized differential equations and the concept of a "principal solution" of such an equation. It is the purpose of the present note to discuss these concepts.

2. SECOND-ORDER SELF-ADJOINT EQUATIONS

Suppose that r, p, m are real-valued functions on a compact interval $[a, b]$, with $r \neq 0$ on this interval, while r, $1/r$ and p are of class $\mathbf{L}^\infty$ and m

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† Professor of Mathematics, the University of Oklahoma, Norman, Oklahoma 73069.

is of bounded variation. If $\mathbf{D}$ denotes the class of real functions η which are absolutely continuous on $[a, b]$ and $\eta' \in \mathbf{L}^2[a, b]$, then the functional

$$J[\eta] = \int_a^b \{r(t)[\eta'(t)]^2 + p(t)\,\eta^2(t)\}\,dt + \int_a^b dm(t)\,\eta^2(t) \tag{2.1}$$

is quadratic on $\mathbf{D}$, and the corresponding Euler differential equation may be written as the system

$$\begin{aligned} &\text{(a)} \qquad -dv(t) + p(t)\,u(t)\,dt + [dm(t)]\,u(t) = 0,\\ &\text{(b)} \qquad u'(t) - [1/r(t)]\,v(t) = 0, \end{aligned} \tag{2.2}$$

where by definition (u, v) is a solution of (2.2) if u is absolutely continuous on $[a, b]$, v is of bounded variation on this interval, (2.2b) is satisfied almost everywhere, and (2.2a) holds in the sense of the Riemann–Stieltjes integral equation

$$v(t) = v(\tau) + \int_\tau^t p(s)\,u(s)\,ds + \int_\tau^t u(s)\,dm(s) \tag{2.3}$$

for $(t, \tau) \in [a, b] \times [a, b]$.

Now suppose that $a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$ is a partition of $[a, b]$, and that m has constant value m_α on the open subinterval $(t_{\alpha-1}, t_\alpha)$, $(\alpha = 1, \dots, k)$. For brevity, let $l_\alpha^- = m(t_\alpha) - m_\alpha$, $l_{\alpha-1}^+ = m_\alpha - m(t_{\alpha-1})$ for $\alpha = 1, \dots, k$ and $l_\beta = l_\beta^+ + l_\beta^- = m_{\beta+1} - m_\beta$ for $\beta = 1, \dots, k-1$. Then for a solution (u, v) of (2.2), Eq. (2.2a) is equivalent to the following conditions:

(i) on each subinterval $(t_{\alpha-1}, t_\alpha)$ the function v is absolutely continuous and satisfies almost everywhere the equation

$$-v'(t) + p(t)\,u(t) = 0; \tag{2.4}$$

(ii) $v(t_\beta^+) - v(t_\beta^-) - l_\beta u(t_\beta) = 0$, $(\beta = 1, \dots, k-1)$,

(iii) $v(t_\alpha) = l_\alpha^- u(t_\alpha) + v(t_\alpha^-)$, $(\alpha = 1, \dots, k)$,
$v(a) = -l_0^+ u(a) + v(a^+)$.

In particular, if $r(t)$ is equal to a positive constant ρ_α on $(t_{\alpha-1}, t_\alpha)$, $(\alpha = 1, \dots, k)$, and $p(t) \equiv 0$, then (u, v) is a solution of (2.2) on $[a, b]$ if and only if u is the continuous polygonal function whose graph joins the successive points $(t_j, u(t_j))$, $(j = 0, 1, \dots, k)$, and the values $u(t_j)$ satisfy the linear second-order difference equations

$$\rho_{\beta+1}\,\frac{u(t_{\beta+1}) - u(t_\beta)}{t_{\beta+1} - t_\beta} - \rho_\beta\,\frac{u(t_\beta) - u(t_{\beta-1})}{t_\beta - t_{\beta-1}} - l_\beta u(t_\beta) = 0, \qquad (\beta = 1, \dots, k-1), \tag{2.5}$$

while $v(t) = \rho_\alpha u'(t) = \rho_\alpha[u(t_\alpha) - u(t_{\alpha-1})]/[t_\alpha - t_{\alpha-1}]$ on $(t_{\alpha-1}, t_\alpha)$, $v(t_\alpha) = l_\alpha^- u(t_\alpha) + \rho_\alpha[u(t_\alpha) - u(t_{\alpha-1})]/[t_\alpha - t_{\alpha-1}]$ for $\alpha = 1,..., k$, and

$$v(a) = -l_0^+ u(a) + \rho_1[u(t_1) - u(a)]/[t_1 - a].$$

Now, as noted in [9; Theorem 2.3], a pair (u, v) is a solution of the generalized differential equation system (2.2) if and only if $(\hat{u}, \hat{v}) = (u, v - mu)$ is a solution in the Carathéodory sense of an associated system of ordinary differential equations, which in this case is

$$\begin{aligned} -\hat{v}' + [p(t) - m^2(t)/r(t)]\hat{u} - [m(t)/r(t)]\hat{v} &= 0, \\ \hat{u}' - [m(t)/r(t)]\hat{u} - [1/r(t)]\hat{v} &= 0. \end{aligned} \tag{2.5}$$

Consequently, results for linear second-order difference equations (2.5) are contained in results on generalized differential equations of the sort defined above, which in turn are subsumed in the theory of ordinary differential equations. In particular, for the self-adjoint system under consideration the oscillation and comparison theorems of Sturmian theory and the concept of a principal solution are immediately available. For this latter concept, the reader is referred to the initial papers of Leighton and Morse [5], Hartman and Wintner [3], Reid [7], and also [4; Chap. X] and [10; Chap. VII, Sect. 5 and Prob. 10, p. 350, in particular].

3. Sturm's Theorem

Suppose that $P_0(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_n \neq 0$, is a polynomial with real coefficients which has no multiple roots. If $P_1(x)$ is the derivative polynomial $P_0'(x)$, then there is uniquely determined an integer k and polynomials $Q_1(x),..., Q_{k-1}(x)$, $P_2(x),..., P_k(x)$ with real coefficients such that for $\alpha = 1,..., k - 1$ the degree of $P_{\alpha+1}(x)$ is less than the degree of $P_\alpha(x)$, and

$$\begin{aligned} P_0(x) &= Q_1(x)\, P_1(x) - P_2(x), \\ P_j(x) &= Q_{j+1}(x)\, P_{j+1}(x) - P_{j+2}(x), \qquad (j = 1,..., k-3), \\ P_{k-2}(x) &= Q_{k-1}(x)\, P_{k-1}(x) - P_k, \end{aligned} \tag{3.1}$$

where P_k is a nonzero constant. The simplest case of Sturm's theorem is as follows: *If $c < d$, and neither c nor d is a zero of $P_0(x)$, then the number of real zeros of $P_0(x)$ on the open interval (c, d) is equal to the excess of the number of variations of signs of the sequence $\{P_0(c), P_1(c),..., P_{k-1}(c), P_k\}$ over the number of variations of signs of the sequence $\{P_0(d), P_1(d),..., P_{k-1}(d), P_k\}$.*

Now for $x \in (-\infty, \infty)$, define $u(j) = u(j; x)$, $(j = 0, 1, \ldots, k)$, as $u(j) = P_j(x)$. Then $\{u(0), u(1), \ldots, u(k)\}$ satisfies the system of difference equations

$$(\Delta) \quad u(\beta + 1) - 2u(\beta) + u(\beta - 1) - l_\beta u(\beta) = 0, \qquad (\beta = 1, \ldots, k - 1),$$

where $l_\beta = l_\beta(x)$ is given by $l_\beta = Q_\beta(x) - 2$. Moreover, if we define

$$\begin{aligned} l_k(x) &= [P_{k-1}(x)/P_k] - 1, \\ l_\alpha(x) &= 0, \qquad \text{for} \quad \alpha = k + 1, k + 2, \ldots \quad \text{and} \quad \alpha = 0, -1, -2 \ldots, \end{aligned} \tag{3.2}$$

then the difference system

$$(\Delta^\infty) \quad u(i + 1) - 2u(i) + u(i - 1) - l_i u(i) = 0, \qquad (i = 0, \pm 1, \pm 2, \ldots)$$

has as a solution $u(j) = P_j(x)$, $(j = 0, 1, \ldots, k)$, and

$$\begin{aligned} u(i) &= P_k \qquad \text{for} \quad i = k + 1, k + 2, \ldots \\ u(i) &= P_0(x)[1 - i] + iP_1(x), \qquad \text{for} \quad i = -1, -2, \ldots . \end{aligned} \tag{3.3}$$

That is, if $u(t) = u(t; x)$ is the solution of the associated generalized differential system (2.2) determined by the above defined solution $u(\alpha) = u(\alpha; x)$ of (Δ), then $u(t) \equiv P_k$ on $k \leqslant t < \infty$, and $u(t; x)$ is linear in t on $(-\infty, 1]$. For brevity, this associated system (2.2) will be denoted by $(2.2_{\Delta\infty})$.

Moreover, from the fact that $l_j = 0$ for $j > k$ it follows that this generalized differential system is disconjugate for large t, and that the thus determined solution $u(t) = u(t; x)$ is the principal solution at ∞ of $(2.2_{\Delta\infty})$. Consequently, the zeros of $u(t; x)$ are the values which are conjugate to ∞ with respect to $(2.2_{\Delta\infty})$, and in view of the linearity of $u(t; x)$ in t on $(-\infty, 1)$ at most one such zero is on $(-\infty, 0)$. Moreover, a zero of $u(t, x)$ on $(0, k)$ corresponds to a variation of sign in the sequence $\{P_0(x), P_1(x), \ldots, P_{k-1}(x), P_k\}$. Therefore, the result of Sturm's theorem is a direct consequence of the fact that if x_1 is a real value such that $u(0; x_1) = P_0(x_1) = 0$, then for x in a suitably small neighborhood N of x_1 there is a zero $t = T(x)$ of $u(t; x) = 0$ which is such that $T(x_1) = 0$, $T(x)$ is continuous and decreasing on N, so that as x decreases through the value x_1 the zero $T(x)$ is moving to the right. This latter argument, which is an immediate consequence of the linearity of $u(t; x)$ in t on $(-\infty, 1]$, and the fact that $P_0(x)$ and $P_0'(x)$ are not both zero for any x, is the usual argument for establishing Sturm's theorem. The point to be emphasized here is that the above discussion shows the precise relation of this theorem to the oscillation theory for the generalized differential system which is associated with the sequence of polynomials occurring in the proof of Sturm's theorem.

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